Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

A THEOREM ON CROSS–RATIOS IN THE GEOMETRY OF INVERSION.

By J. L. Walsh.

It is the purpose of this paper to present a solution of the following problem. Let C_1, C_2, C_3, C_4 be four fixed distinct non-null circles in the plane. Let z_1, z_2, z_3, z_4 be the inverses of a variable point z regarding these circles respectively. What are the geometrical characteristics of the configuration of these four circles if the cross-ratio (z_1, z_2, z_3, z_4) is a constant independent of the position of the point z? A complete answer to this question is given in Theorem III below.

As is usual in the geometry of inversion, we adjoin to the finite plane a single point at infinity, and we use the term *circle* to include straight lines as well as circles in the ordinary sense of the word. In proving Theorem III we shall give several preliminary theorems.

Theorem I. *Let α_1, α_2, α_3, α_4 be any four fixed non-concyclic points of the plane. Denote by C_i $(i = 1, 2, 3, 4)$ the circle passing through the three of these points obtained by omitting α_i; and denote by z_i the inverse of a variable point z with respect to C_i. Then the cross-ratio (z_1, z_2, z_3, z_4) does not depend on the position of the point z but is constantly equal to $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.**

By means of a linear transformation, transform α_1 to infinity, α_4 to the point 1, and the inverse of α_1 with respect to C_1 to the origin. This transformation is always possible. The circle C_1 is the unit circle whose center is the origin, and z_2 and z_3 are points on C_1 distinct from each other and from the point 1. The circle C_2 is the line through 1 and α_3, C_3 is the line through 1 and α_2, C_4 is the line through α_2 and α_3.

The function z_i $(i = 1, 2, 3, 4)$ is a linear function of $\bar{z}$, the conjugate imaginary of z. For we may consider z_i obtained from $\bar{z}$ by reflection in the axis of reals (which brings us to z) followed by reflection in the circle C_i. Successive reflection in two circles is always a linear transformation.

Let us compute z_i in terms of $\bar{z}$. Of course we have $z_1 = 1/\bar{z}$. The other functions z_i are integral functions of $\bar{z}$ since $\bar{z} = \infty$ corresponds to $z_i = \infty$. Moreover $z = z_2$ when $z = 1$ or α_3, $z = z_3$ when $z = 1$ or α_2, $z = z_4$ when $z = \alpha_2$ or α_3. Since $\alpha_2\bar{\alpha}_2 = 1$, $\bar{\alpha}_3\alpha_3 = 1$, we have

$$\begin{cases} z_2 = -\alpha_3\bar{z} + 1 + \alpha_3, \\ z_3 = -\alpha_2\bar{z} + 1 + \alpha_2, \\ z_4 = -\alpha_2\alpha_3\bar{z} + \alpha_2 + \alpha_3. \end{cases}$$

---

* Strictly speaking, there is an exception if $z = \alpha_i$, for the cross-ratio (z_1, z_2, z_3, z_4) is then not defined. The theorem is true in the sense that whenever the cross-ratio (z_1, z_2, z_3, z_4) is defined, it has the value $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. A similar remark applies to Theorems II and III below.

The cross-ratio which we are considering is

$$(z_1, z_2, z_3, z_4) \equiv \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)},$$

which reduces to

$$\frac{\alpha_3 - 1}{\alpha_3 - \alpha_2} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4).$$

This completes the proof.*

It will be noted that the constant cross-ratio referred to in Theorem I is never real. A theorem analogous to Theorem I but referring instead to real cross-ratios is

THEOREM II. *If we denote by z_1, z_2, z_3, z_4 the respective inverses of a variable point z regarding four fixed non-null coaxal circles C_1, C_2, C_3, C_4, then the cross-ratio (z_1, z_2, z_3, z_4) is real and independent of the position of z.*

The four given circles may have two common points, they may all be tangent at a single point, or they may have no common point. We consider these cases in order.

If all four circles have two common points, we transform one of these points to infinity and the other to the origin. The circles C_1, C_2, C_3, C_4 are transformed into straight lines through the origin; we denote by θ_k (where $k = 1, 2, 3, 4$) any angle which the line C_k makes with the axis of reals. The inverse of the point $z = re^{i\varphi}$ with respect to the circle C_k is then $re^{i(2\theta_k - \varphi)}$, and the cross-ratio with which we are concerned is therefore

$$\frac{[re^{i(2\theta_1-\varphi)} - re^{i(2\theta_2-\varphi)}][re^{i(2\theta_3-\varphi)} - re^{i(2\theta_4-\varphi)}]}{[re^{i(2\theta_2-\varphi)} - re^{i(2\theta_3-\varphi)}][re^{i(2\theta_4-\varphi)} - re^{i(2\theta_1-\varphi)}]},$$

which reduces to

$$\frac{[e^{2i\theta_1} - e^{2i\theta_2}][e^{2i\theta_3} - e^{2i\theta_4}]}{[e^{2i\theta_2} - e^{2i\theta_3}][e^{2i\theta_4} - e^{2i\theta_1}]},$$

a number independent of z. In the form in which it is written, this number represents the cross-ratio of the four inverses of $z = 1$. These all lie on the unit circle whose center is the origin and hence their cross-ratio is real.

---

* We indicate briefly another proof of Theorem I. Suppose the configuration transformed as previously indicated. The function $(z_1 - z_2)(z_3 - z_4)/(z_2 - z_3)(z_4 - z_1)$ is a rational function of $\bar{z}$. It can become infinite only when $z_2 = z_3$ or $z_4 = z_1$, that is, when $z = \alpha_1$, α_2, α_3, or α_4. For definiteness consider the point α_3, and allow z to lie on the circle C_1 and to approach the point $z = \alpha_3$. The quotient $(z_3 - z_4)/(z_3 - z_2)$ approaches unity, the quotients $(z_1 - z_2)/(z_1 - \alpha_3)$, $(z_1 - \alpha_3)/(z_1 - z_4)$ approach finite limits and hence the rational function does not become infinite at $z = \alpha_3$. Similarly it can be proved not to become infinite at any of the points $z = \alpha_1$, α_2, or α_4, and hence is a constant. It remains to evaluate the constant.

Let $z = 0$, so that $z_1 = \infty$. The points $z_2 = 1 + \alpha_3$, $z_3 = 1 + \alpha_2$, $z_4 = \alpha_2 + \alpha_3$ are the vertices of a triangle congruent to the triangle whose vertices are α_2, α_3, $\alpha_4 = 1$. Then the cross-ratio $(z_1, z_2, z_3, z_4) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ for $z = 0$ and hence for all values of z.

If the four original circles are all tangent at a single point, we transform that point to infinity and the circles into lines parallel to the axis of imaginaries. The circles C_1, C_2, C_3, C_4 will be lines $x = a_1, a_2, a_3, a_4$ respectively. The inverse of a point $z = x + iy$ with regard to the circle C_k is the point $(2a_k - x) + iy$. The cross-ratio of the four inverses is

$$\frac{(a_1 - a_2)(a_3 - a_4)}{(a_2 - a_3)(a_4 - a_1)},$$

which is not only real and independent of the position of z but is also the cross-ratio of the points in which the lines are cut by any transversal.

If the four original circles have no point in common, there are two null circles of the coaxal family. Transform one of these to infinity and the other to the origin, so that C_1, C_2, C_3, C_4 become circles whose common center is the origin; we denote their respective radii by r_1, r_2, r_3, r_4. The inverse of $z = re^{i\varphi}$ with regard to C_k is $r_k^2 e^{i\varphi}/r$, and the cross-ratio of the four inverses reduces to

$$\frac{(r_1^2 - r_2^2)(r_3^2 - r_4^2)}{(r_2^2 - r_3^2)(r_4^2 - r_1^2)},$$

which is real and independent of z. The proof of Theorem II is thus complete.

Suppose now we have four distinct fixed non-null circles C_1, C_2, C_3, C_4, and that the cross-ratio (z_1, z_2, z_3, z_4) of the four inverses of a point z is a constant independent of z. We shall prove that we have a situation such as appears either in Theorem I or Theorem II.

The cross-ratio can have none of the degenerate values 0, 1, ∞. Suppose for definiteness that it has the constant value zero. Then either $z_1 = z_2$ or $z_3 = z_4$ for an infinite number of values of z. From the reasoning previously used, z_1 is a linear function of z_2 and z_3 is a linear function of z_4. Hence we must have $z_1 \equiv z_2$ or $z_3 \equiv z_4$, which means that C_1 coincides with C_2 or C_3 coincides with C_4; either of these suppositions is contrary to our hypothesis.

If any two of the circles C_1, C_2, C_3, C_4, say for definiteness C_1 and C_2, have a point α in common, a third circle of the set must pass through α. For we may choose $z = \alpha$, so that $z_1 = z_2 = \alpha$. If the cross-ratio is not to have the value zero we must have either $z_2 = z_3 = \alpha$ or $z_4 = z_1 = \alpha$; a point coincides with its inverse only when it is on the circle of inversion, so C_3 or C_4 must pass through α.

If any two of the original four circles, for definiteness C_1 and C_2, have no point in common, a third circle of the set is coaxal with them. For there exist two points α and β mutually inverse regarding both C_1 and C_2. Let $z = \alpha$ and we have $z_1 = z_2 = \beta$. Hence we must have $z_2 = z_3 = \beta$

or $z_4 = z_1 = \beta$; that is, the inverse of α with respect to C_3 or C_4 is β. Then C_3 or C_4 is coaxal with C_1 and C_2.

If two of the circles C_1, C_2, C_3, C_4 have no point in common, all four circles are coaxal. For we have just shown three—for definiteness C_1, C_2, C_3—to be coaxal; no two of these three circles can have a point in common. Then C_4 can have no point in common with any of the circles C_1, C_2, C_3. If it has no point in common with C_1, the circles C_1, C_2, C_4 or C_1, C_3, C_4 must be coaxal. Hence all four circles are coaxal.

If two, say C_1 and C_2, of the original four circles are tangent at a point α, all four circles are mutually tangent at α. If C_3 is not tangent to C_1 and does not pass through α, it must cut C_1 in two points distinct from α. Then C_4 must pass through α and through these two points and hence coincide with C_1. Therefore each of the circles C_3 and C_4 must either pass through α or be tangent to both C_1 and C_2 in points distinct from α. Both C_3 and C_4 cannot pass through α unless both are tangent at α to C_1 and C_2; both C_3 and C_4 cannot be tangent to C_1 and C_2 in points distinct from α. Suppose for definiteness that C_4 does not pass through α but is tangent to C_1 and C_2 respectively in points β and γ distinct from α. Then C_3 must pass through α, β, and γ. Transform α, β, γ to ∞, $+1$, -1, respectively. Then C_3 becomes the axis of reals, C_4 becomes the unit circle whose center is the origin, and C_1 and C_2 become the lines tangent to C_4 at $+1$ and -1 respectively. These four circles do not satisfy the hypothesis we have made. For when z is on C_3, all the points z_1, z_2, z_3, z_4 are also on C_3 and hence their cross-ratio is real. On the other hand, if z is not on C_3 but is on C_1, z_4 is interior to the triangle formed by z_1, z_2, z_3, the four points are not concyclic, their cross-ratio is not real and therefore not constant.

If two of the circles C_1, C_2, C_3, C_4, say C_1 and C_2, have two distinct points α and β in common, the four circles either are coaxal or form a configuration such as that described in Theorem I. For either C_3 or C_4, say C_3, must pass through α. If C_3 passes through β as well, C_4 must pass through both α and β and hence the statement is proved. If C_3 passes through α but not through β, it intersects C_1 and C_2 respectively in points γ and δ distinct from each other and from α and β. Then C_4 must pass through β, γ, and δ, so we have the kind of configuration described in Theorem I. This completes the proof of

THEOREM III. *Let C_1, C_2, C_3, C_4 be four distinct fixed non-null circles. Denote by z_1, z_2, z_3, z_4 the inverses of a variable point z with regard to these four circles respectively. A necessary and sufficient condition that the cross-ratio (z_1, z_2, z_3, z_4) be real and independent of the position of z is that C_1, C_2, C_3, C_4 be coaxal. A necessary and sufficient condition that the cross-ratio* *be non-real and independent of the position of* z *is that the four circles pass by threes through four distinct points. If we denote by* α_i *the point through which pass the three circles which do not include* C_i, *we shall have**

$$(z_1, z_2, z_3, z_4) \equiv (\alpha_1, \alpha_2, \alpha_3, \alpha_4).$$

In Theorem III we have supposed none of the circles C_1, C_2, C_3, C_4 to be a null circle. We shall now consider the possibility that some or all of these may be null circles, but shall suppose that all four circles are distinct. It follows as before that the cross-ratio does not degenerate.

If we choose any four points of the plane, α_1, α_2, α_3, α_4, consider them as null circles, and consider the inverse of a point z with regard to α_i to be the point α_i itself, of course the cross-ratio (z_1, z_2, z_3, z_4) is constantly $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. Three of the four original circles cannot be null circles unless the fourth is also a null circle. For three of the points z_1, z_2, z_3, z_4 and a constant cross-ratio determine uniquely the fourth of those points, which is therefore fixed independent of z.

Suppose two of the original circles, for definiteness C_1 and C_2, are null circles while the other two are non-null circles. We consider in detail the possibilities that C_3 and C_4 have two points in common, are tangent, or have no point in common. If C_3 and C_4 have two points in common, the proof formerly given shows that C_1 and C_2 must lie at the two intersections of C_3 and C_4. Transform C_1 and C_2 to the origin and to infinity respectively, and denote by θ_3 and θ_4 the respective angles which C_3 and C_4—now straight lines through the origin—make with the axis of reals. If we choose any point $z = r^{i\varphi}$, the corresponding inverses are $z_1 = 0$, $z_2 = \infty$, $z_3 = re^{i(2\theta_3-\varphi)}$, $z_4 = re^{i(2\theta_4-\varphi)}$. The cross-ratio is

$$(z_1, z_2, z_3, z_4) = \frac{\epsilon^{2i\theta_4} - \epsilon^{2i\theta_3}}{\epsilon^{2i\theta_4}},$$

which is independent of the position of the point z.

If C_3 and C_4 are tangent, their point of tangency must be either C_1 or C_2, say for definiteness C_1. Transform C_1 to infinity, C_2 to the origin, and C_3 and C_4 into lines parallel to the axis of imaginaries. When z is real, its four inverses are concyclic and hence their cross-ratio is real. When z is not real, the four inverses are not concyclic and their cross-ratio is not real and hence not constant.

If C_3 and C_4 have no point in common, the proof formerly given shows that C_1, C_2, C_3, C_4 are coaxal. Hence C_1 and C_2 are the null circles of the coaxal family determined by C_3 and C_4. The reader can easily compute the cross-ratio of the four inverses of z and show that it is independent of z.

---

* Part of the proof of so much of Theorem III as refers to the necessity of the condition was worked out jointly by Professor J. L. Coolidge and myself. Theorem IV was suggested to me by Professor Coolidge.

We give now the results if one and only one of the original circles is a null circle. The proofs are so similar to the foregoing that they are omitted. If two of the original circles have two points in common, a third is coaxal with them and the fourth is a point common to them all. If two of the circles are tangent, a third is coaxal with them and the fourth is the common point of tangency. If two of the circles have no point in common, a third is coaxal with them and the fourth is a null circle of that coaxal family. In all of these cases the cross-ratio (z_1, z_2, z_3, z_4) is a constant independent of the position of the point z.

A theorem closely connected with the first part of Theorem III is the following:

THEOREM IV. *Let there be given four distinct fixed non-null circles in the plane. Denote by* z_1, z_2, z_3, z_4 *the inverses in these circles respectively of a variable point* z *of the plane. A necessary and sufficient condition that* z_1, z_2, z_3, z_4 *be concyclic whatever be the position of* z *is that the four given circles be coaxal.*

The necessity of the condition is easily proved by methods somewhat similar to those previously used. Denote the given circles by C_1, C_2, C_3, C_4 respectively. Choose z on C_1 but on none of the other circles; then z coincides with z_1. The circle C through z_1, z_2, z_3, z_4 passes through three pairs of distinct points mutually inverse regarding C_2, C_3, C_4, and hence C is orthogonal to C_2, C_3, C_4. We can choose z on C_1 but on none of the circles C_2, C_3, C_4 in an infinite variety of ways and hence we have either an infinity of circles C orthogonal to C_2, C_3, C_4 in which case these three circles are coaxal or we have C coinciding with C_1, so that C_1 is orthogonal to C_2, C_3, C_4. Similarly, we can of course choose C_2, C_3, or C_4 instead of C_1 and prove for example that either C_1, C_3, C_4 are coaxal or C_2 is orthogonal to them all.

If any three of the four circles C_1, C_2, C_3, C_4, say for definiteness C_2, C_3, C_4, are coaxal, all four circles are coaxal. For we know that either C_1, C_3, C_4 are coaxal or C_2 is orthogonal to them all. Since C_2 is coaxal with C_3 and C_4 it is not orthogonal to both C_3 and C_4. Hence C_1, C_3, C_4 are coaxal and therefore all four circles are coaxal.

If no set of three of the four original circles is coaxal, each of those four circles is orthogonal to the other three, which is of course impossible. In fact two of the circles are easily transformed into two perpendicular lines; a third circle must have its center at their intersection; there is evidently no fourth circle orthogonal to all three.

The necessity of the condition of Theorem IV has thus been proved; its sufficiency follows from the reality of the cross-ratio in Theorem II and hence completes the proof. In Theorem IV the four points z_1, z_2,

z_3, z_4 are not only concyclic but are concyclic with z. This follows from inspection of the proof of Theorem II. It immediately suggests

THEOREM V. *Let there be given three distinct fixed non-null circles in the plane. Denote by* z_1, z_2, z_3 *the inverses in these circles respectively of a variable point* z *of the plane. A necessary and sufficient condition that* z, z_1, z_2, z_3 *be concyclic whatever be the position of* z *is that the three given circles be coaxal.*

First we prove the necessity of the condition. Through any point z not on one of the given circles and through its inverses z_1, z_2, z_3 there passes a circle which is orthogonal to the three given circles. Hence those circles are coaxal.

The sufficiency of the condition follows easily by the method of proof of Theorem II.

HARVARD UNIVERSITY.